

The University of Chicago

TWO NEW UNIVERSAL WARING THEOREMS

A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF DOCTOR
OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

1938

By

MARY BARBARA HABERZETLE

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THE WARING PROBLEM WITH SUMMANDS x^m , $m \geq n$

BY MARY HABERZETLE

In the original Waring problem the summands considered are positive integral n -th powers. By using different types of summands one obtains various modifications of the problem. In this paper those of the form x^m , $m \geq n$, are used. If a positive integer is a sum of u such summands with $x \geq 0$, it is said to be a sum of u "values". If $n \geq 9$ and $r \leq 2^n - k - 3$, $I = 2^n + k - 1$ values are necessary and sufficient to represent every positive integer; r is defined by $3^n = 2^n q + r$, $q = [(\frac{3}{2})^n]$, and $k = [.584963n]$. The notation $[x]$ denotes the largest integer $\leq x$. For $9 \leq n \leq 400$ it is shown that the inequality $r \leq 2^n - k - 3$ is satisfied.

This problem was suggested by L. E. Dickson.

1. Representation of integers $\leq 2^n q$. Write $3^n = 2^n q + r$, $1 \leq r < 2^n$, $q = [(\frac{3}{2})^n]$.

LEMMA A. *Every positive integer $< 2^x - 1$ is a sum of at most $x - 1$ terms $1, 2, \dots, 2^{x-1}$.*

The lemma may be verified for $x = 1$, $x = 2$. Assume, therefore, that it is true for integers $< 2^x - 1$ and prove by induction. The integer $2^x - 1$ is equal to $1 + 2 + \dots + 2^{x-1}$ and hence requires x terms. The integers $2^x + 1$, $2^x + 2, \dots, 2^{x+1} - 2$ are each the sum of 2^x and an integer $< 2^x - 1$. Hence all $< 2^{x+1} - 1$ are sums of at most x terms $1, 2, \dots, 2^x$.

Let k be the maximum positive integer such that $2^{n+k} \leq 2^n q$. Then k is the largest integer for which $2^k \leq q = [(\frac{3}{2})^n]$, that is,

$$k = [(n \log \frac{3}{2}) / \log 2] = [.584963n].$$

LEMMA 1. *All positive integers $\leq 2^n q$ are sums of $I = 2^n + k - 1$ values. The integer $2^{n+k} - 1$ requires exactly I values.*

Let $B = 2^n x + y$, $0 \leq y \leq 2^n - 1$, $0 \leq x < q$. Then $x < 2^{k+1} - 1$ and by Lemma A is a sum of at most k terms $1, 2^2, \dots, 2^k$. Hence B is a sum of $2^n + k - 1$ values. Since $q \leq 2^{k+1} - 1$, $2^n q$ is a sum of $k + 1$ values. The integer $2^{n+k} - 1$ is equal to $2^n(1 + 2 + \dots + 2^{k-1}) + 2^n - 1$ and hence is a sum of $2^n + k - 1$ values and not fewer.

2. Results from the asymptotic theory. The following theorem, whose proof is due to Vinogradov¹ and L. E. Dickson,² is stated here with an improved

Received June 9, 1938.

¹ Vinogradov, *On Waring's problem*, Annals of Mathematics, vol. 36(1935), pp. 395-405.

² L. E. Dickson, *Proof of the ideal Waring theorem for exponents 7-180*, American Journal of Mathematics, vol. 58(1936), p. 525.

value for N . The improvement is due to some later refinements³ made in the theory by Dickson.

THEOREM 1. *Let k_0 be the least integer exceeding $-\log r_1/[\log(1-\nu)]$, where*

$$r_1 = \frac{n^2(6n-1)}{n-d}, \quad d = 1 + 2n^2z, \quad z = \frac{1}{12}\nu^3, \quad \text{and} \quad \nu = n^{-1}.$$

Take $k_1 = 2k_0$, $\log_{10} N = .8n^6$. If $n \geq 9$, every integer $\geq N$ is a sum of $4n - 2 + 3k_1$ integral n -th powers ≥ 0 .

3. Ascent and solution for $9 \leq n \leq 400$. Since $4n - 2 + 3k_1 < 2^n + k$, it follows from Theorem 1 and Lemma 1 that all positive integers $\leq 2^n q$ and $\geq N$ are sums of $I = 2^n + k - 1$ values. The following theorem will be proved by ascent.

THEOREM 2. *For $9 \leq n \leq 400$ all positive integers are sums of $I = 2^n + k - 1$ values.*

The following lemmas are required.

LEMMA 2. *All integers in the interval $(2^n q, 2^n q + 2^n)$ are sums of E values, where $E = \max(k+r, 2^n-r)$.*

Since $2^n q$ is a sum of $k+1$ values, $2^n q, 2^n q+1, \dots, 2^n q+r-1$ are sums of $k+r$ values. The integers $2^n q+r=3^n, \dots, 2^n q+2^n-1=3^n-r+2^n-1$ are sums of 2^n-r values. Also $q+1 \leq 2^{k+1}$. Hence $2^n q+2^n$ is a sum of $k+1$ values.

The interval in Lemma 2 can be extended to $(2^n q, 2^n q + 2^n q)$ by adding on multiples of 2^n , that is, $2^n x, 1 \leq x < q$. By application of Lemma A it is seen that at most k additional values are required to make this extension. Hence all in $(2^n q, 2^n q + 2^n q)$ are sums of $E+k$ values. As in the preceding paragraph the integers in $(2^n q + 2^n q, 2^n q + 2^n q + r - 1)$ can be shown to be sums of $k+r$ and hence of E values. The integer $2^n q + 2^n q + r = 2^n q + 3^n$ is a sum of $k+2 < E+k$ values. This proves

LEMMA 3. *All integers in $(2^n q, 2^n q + 3^n)$ are sums of $E+k$ values.*

LEMMA⁴ 4. *Let L, p, n, z, w be positive integers. If all integers in $(L, L + p^n z)$ are sums of m values, then all in $(L, L + p^n(z+w))$ are sums of $m+w$ values.*

This with $z=1$ and Lemma 3 give

LEMMA 5. *All integers in $(2^n q, 2^n q + 3^n(1+w))$ are sums of $E+k+w$ values.*

Take $w = [(\frac{4}{3})^n]$. Then $1+w > (\frac{4}{3})^n$. This proves

LEMMA 6. *All integers in $(2^n q, 2^n q + 4^n)$ are sums of $E+k + [(\frac{4}{3})^n]$ values.*

By induction one obtains

³ L. E. Dickson, *Universal forms $\sum a_i x_i^n$ and Waring's problem*, Acta Arithmetica, vol. 2 (1937), pp. 186-191.

⁴ Dickson, *Proof of the ideal Waring theorem for exponents 7-180*, op. cit., p. 526.

LEMMA 7. All integers in the interval $(2^n q, 2^n q + (n+1)^n)$ are sums of $E + k + [(\frac{4}{3})^n] + [(\frac{5}{4})^n] + \dots + [(n+1)/n]^n$ values, and hence are sums of $E + k + (n-2)[(\frac{4}{3})^n]$ values.

LEMMA⁵ 8. Let $v = (1 - l/L_0)/n$, $v^n L_0 \geq 1$. If all integers between l and L_0 inclusive are sums of Q values, then all between l and L_t inclusive are sums of $Q + t$ values, where

$$(1) \quad \log_{10} L_t = \left(\frac{n}{n-1} \right)^t (\log_{10} L_0 + n \log_{10} v) - n \log_{10} v.$$

For N in Theorem 1 it is desired that $L_t > N$. Hence t is to be determined such that

$$(2) \quad \left(\frac{n}{n-1} \right)^t (\log_{10} L_0 + n \log_{10} v) - n \log_{10} v > .8n^6.$$

Take $l = 3^n$ and $L_0 = (n+1)^n$. Then the interval (l, L_0) is included in the interval of Lemma 7. Also, the inequality $v^n L_0 \geq 1$ reduces to $v(n+1) \geq 1$ and hence to $(\frac{1}{3}(n+1))^{n-1} \geq 3$, an inequality which holds if $n \geq 4$.

Case $n = 9$. Here $\log_{10} L_0 = 9$, $\log_{10} v = \bar{1}.0457488$. Increase $.8n^6$ to n^6 in (2). Then $t = 120$. Also $k = 5$, $r = 227$. Hence $E = \max(k + r, 2^n - r) = 285$. By Lemma 7 and the preceding all integers between $2^n q$ and N inclusive are sums of

$$(3) \quad t + E + k + (n-2)[(\frac{4}{3})^n]$$

values. Since $[(\frac{4}{3})^9] = 13$, (3) is equal to $501 < I = 516$.

Case $n = 10$. Now $\log_{10} L_0 = 10.413927$, $\log_{10} v = \bar{2}.9999990$ and from (2), $t = 140$. From $k = 5$, $r = 681$, it follows that $E = 686$. Also $[(\frac{4}{3})^{10}] = 17$. Hence (3) is equal to $967 < I = 1028$.

When $n \geq 11$, $v = n^{-1}$ to seven decimal places. From the series for $\log(1+x)$, $\log_{10} L_0 - n \log_{10} n$ is the product of n by $\log_{10}(1 + n^{-1}) > M(n^{-1} - \frac{1}{2}n^{-2})$, $M = .4343$. Then $L_t > N$ if

$$(4) \quad \left(\frac{n}{n-1} \right)^t M(1 - \frac{1}{2}n^{-1}) > n^6, \quad n \geq 11.$$

If the denominator $2n$ is replaced by 22 , (4) will follow from

$$t \log_{10} \frac{n}{n-1} > .3824138 + 6 \log_{10} n.$$

The coefficient of t is the product of M by

$$\log_e \left(1 + \frac{1}{n-1} \right) > \frac{1}{n-1} - \frac{1}{2(n-1)^2} > \frac{1}{n}, \quad n > 2.$$

⁵ L. E. Dickson, *Recent progress on Waring's theorem and its generalization*, Bulletin of the American Mathematical Society, vol. 39(1933), p. 711.

Hence $L_t > N$ if

$$(5) \quad t > M^{-1}n(.3824138 + 6 \log_{10} n), \quad n \geq 11.$$

In (3) first let $E = 2^n - r$. Then (3) is $\leq I = 2^n + k - 1$ if

$$(6) \quad \frac{r}{2^n} \geq F(n), \quad F(n) = \frac{t+1}{2^n} + \frac{n-2}{q}.$$

Next let $E = k + r$. The like conclusion holds if

$$(7) \quad \frac{r}{2^n} \leq 1 - F(n) - \frac{k}{2^n}.$$

Cases $n = 11, 12, 13$. Let $n = 11$. From (5), $t = 168$. Then $F(n) = .187170$ and $.187170 \leq r/2^n \leq .809900$. Since $F(n)$ decreases when n increases, Theorem 2 will hold for any $n \geq 11$ for which $r/2^n$ lies between these decimals. Note that $r/2^n$ is the decimal part of $(\frac{3}{2})^n$. The condition holds for $n = 11, 12, 13$ but not for $n = 14$.

Cases $14 \leq n \leq 28$. When $n = 14$ the decimal in (5) reduces to .3780043, and it suffices to take $t = 234$. $F(n) = .055580$. Hence $.055580 \leq r/2^n \leq .943932$. This condition is satisfied for $n = 14, 15, \dots, 28$.

Cases $29 \leq n \leq 162$. Let $n = 20$. The decimal part in (5) is reduced to .3732056. Take $t = 377$. Then $.005773 \leq r/2^n \leq .994216$. This condition is satisfied for $n = 20, \dots, 162$.

Cases $163 \leq n \leq 400$. Let $n = 100$. The decimal in (5) is reduced to .3643871. Take $t = 2847$. $F(n)$ begins with at least 14 zeros, $k/2^n$ begins with 27 zeros, and hence (7) with at least 14 nines. The decimal $r/2^n$ lies between these limits at least as far as $n = 400$.

4. Solution for $n \geq 35$. The results of this section together with the preceding will show that for $n \geq 9$ every positive integer is a sum of I values except when $r \geq 2^n - k - 2$. It can be seen from the preceding section that the inequality $r \geq 2^n - k - 2$ does not hold for $9 \leq n \leq 400$. Whether the inequality is ever satisfied has not yet been determined.

Let $q = [(\frac{3}{2})^n]$, $f = [(\frac{4}{3})^n]$, $g = [(\frac{5}{4})^n]$, and $s = f + 2g$.

LEMMA 9. *If all integers in the interval $(L, L + 2^n)$ are sums of m values, then all in $(L, L + (n + (n + 1)^n))$ are sums of*

$$m + k + 1 + f + g + [(\frac{6}{5})^n] + \dots + \left[\left(\frac{n+1}{n} \right)^n \right]$$

values.

The interval $(L, L + 2^n)$ can be extended to $(L, L + 3^n)$ by adding on multiples of 2^n , that is, $2^n x$, $1 \leq x \leq q$. The final integer so obtained is $L + 2^n(q + 1) > L + 3^n$. The integer x is a sum of at most $k + 1$ terms $1, 2, \dots, 2^k$. Hence in ascending to $L + 3^n$ at most $k + 1$ additional values are required. To complete the proof of Lemma 9 apply Lemma 4 as in the proof of Lemma 7.

LEMMA 10. If $n \geq 35$, $L \leq 3 \cdot 4^n$, and all integers in the interval $(L, L + 2^n)$ are sums of m values, then every integer $\geq L$ is a sum of $m + s + k - 1$ values.

Since $4n - 2 + 3k_1$ is less than $s + k$ when $n \geq 35$ and hence less than $m + s + k$, it follows from Theorem 1 that all integers $\geq N$ are sums of $m + s + k - 1$ values. It will be proved that all integers $\geq L$ and $\leq N$ are sums of $m + s + k - 1$ values by ascent. By applying Lemma 8 for $n \geq 35$ as in §3 for $n \geq 11$, but now with $l = 3 \cdot 4^n$, $L_0 = (n + 1)^n$, and Lemma 9 instead of Lemma 7, it suffices to show that

$$(8) \quad t + 2 + \left[\left(\frac{9}{5}\right)^n\right] + \cdots + \left[\left(\frac{n+1}{n}\right)^n\right] \leq \left[\left(\frac{5}{4}\right)^n\right]$$

where $t > M^{-1}n(3684214 + 6 \log_{10} n)$.

Multiply (8) by $(\frac{4}{5})^n$. The resulting inequality need be verified only for $n = 35$ since np^n , $np^n \log_{10} n$, and p^n decrease when n increases and $n \geq 7$, $0 < p \leq \frac{4}{5}$. Take $t = 777$. Then

$$\left(\frac{4}{5}\right)^n \left(t + 2 + \left[\left(\frac{9}{5}\right)^n\right] + \cdots + \left[\left(\frac{n+1}{n}\right)^n\right] \right) < .9224296 < \left[\left(\frac{5}{4}\right)^n\right] \left(\frac{4}{5}\right)^n, \quad n \geq 35$$

Hence (8) is satisfied.

LEMMA 11. If $n \geq 35$ and $s \leq r \leq 2^n - k - s$, every positive integer is a sum of I values.

The integers $2^n q, \dots, 2^n q + r - 1$ are sums of $k + r \leq 2^n - s$ values. The integers $\geq 2^n q + r = 3^n$ and $< 2^n(q + 1)$ and hence $\leq 3^n + 2^n - r - 1$ are sums of $2^n - r \leq 2^n - s$ values. Finally, $2^n(q + 1)$ is a sum of $k + 1$ values. Apply Lemma 10 with $L = 2^n q$, $m = 2^n - s$ to obtain the result that every integer $\geq 2^n q$ is a sum of I values. Apply Lemma 1.

The following lemma is required in the proof of Lemma 13.

LEMMA 12. If $1 \leq w \leq (2^n - 1)r^{-1}$, all integers in $(3^n w, 3^n w + 3^n)$ are sums of M_1 values, where M_1 is the maximum of $A = 2^n - wr + k + w - 1$ and $B = w(k + 1 + r) + k$.

Those integers $\leq w3^n + 2^n - wr - 1$ are sums of $w + 2^n - wr - 1$ values. The next integer is equal to $2^n(qw + 1)$. To this add $1, 2, \dots, wr - 1$ and obtain $3^n w + 2^n - 1$ as the final integer. Therefore all those $\geq 2^n(qw + 1)$ and $\leq 3^n w + 2^n$ are sums of $w(k + 1 + r)$ values.

If we add $2^n x$, $1 \leq x < q$, to each integer in $(3^n w, 3^n w + 2^n)$, the interval is extended to $3^n w + 2^n q$. At most k additional values are required in making the extension. Hence all in $(3^n w, 3^n w + 2^n q)$ are sums of M_1 values. The integers in $(3^n w + 2^n q, 3^n w + 3^n)$ require at most $w + k + r < B$ values.

LEMMA 13. If $n \geq 35$ and $r < s$, every positive integer is a sum of I values.

Since 3 belongs to the exponent 2^{n-2} modulo 2^n , $r = 1$ would imply that n is divisible by 2^{n-2} , whence $n \leq 4$. If $r = 3$, then $3^{n-1} = (\frac{1}{3}2^n q) + 1$ and $n - 1 \leq 2$. Hence $r \geq 5$.

For $x = 0, 1, \dots, u(3^n - 2^n q) - 1 = ur - 1$, $2^n qu + x$ is a sum of $(k + 1)u + x \leq (k + 1)u + ur - 1$ values. If $2^n > ur$, there exist integers $\geq 3^n u$ and $\leq 2^n qu + 2^n - 1 = 3^n u + 2^n - 1 - ur$ and each is a sum of $u + 2^n - 1 - ur$ values. The latter and $(k + 1)u + ur - 1$ will be $\leq 2^n - s$ if

$$(9) \quad \frac{s}{r-1} \leq u \leq \frac{2^n - s}{k+1+r}.$$

Decrease r to 5 on the left and increase r to s on the right to obtain the subinterval,

$$(10) \quad \frac{s}{4} \leq u \leq \frac{2^n - s}{k+1+s}.$$

Thus any u satisfying this inequality will satisfy (9). Since $k + 1 + s > s > r$,

$$\frac{2^n - s}{k+1+s} < \frac{2^n}{r}.$$

Hence, $2^n > ur$.

There will exist an integer u satisfying (10) if the difference between the limits is ≥ 1 , and therefore, if $2^n \geq R$,

$$R = 2s + \frac{1}{4}s^2 + (1 + \frac{1}{4}s)(k + 1).$$

But $k + 1 \leq .584963n + 1$ and $s < 3(\frac{4}{3})^n$. Hence $2^n \geq R$ if

$$4 \cdot 2^n \geq 24(\frac{4}{3})^n + 9(\frac{16}{9})^n + 4(k + 1) + 3(\frac{4}{3})^n(k + 1).$$

Multiply both sides by $(\frac{9}{16})^n$. Then

$$4(\frac{9}{8})^n \geq 9 + 24(\frac{3}{4})^n + 4(\frac{9}{16})^n(k + 1) + 3(\frac{3}{4})^n(k + 1).$$

Since the right side decreases as n increases, it suffices to verify the inequality for a particular $n \leq 35$. For $n = 20$ it becomes $42.196 > 9$,⁺ and hence holds for $n \geq 20$.

Thus all integers in $(2^n qu, 2^n qu + 2^n)$ are sums of $2^n - s$ values. By Lemma 10 every integer $\geq 2^n qu$ is a sum of I values.

It remains to show that every positive integer $< 2^n qu$ is a sum of I values. By Lemma 1 and the proof of Lemma 2 this is true for integers $\leq 3^n$. It remains then to prove it for integers between 3^n and $3^n U$, where U is the least integer satisfying (10). Lemma 12 is made use of.

For $w \leq U - 1$, it can be shown that $B \leq A$. Hence, apply Lemma 12 with M_1 taken equal to A . If $2^n - wr + k + w \leq 2^n + k$, and hence if $r \geq 1$, $A \leq I$. Therefore all integers in $(3^n w, 3^n(w + 1))$ are sums of I values. Let $w = 1, 2, \dots, U - 1$. The proof of Lemma 13 is now complete.

LEMMA 14. If $n \geq 35$ and $2^n - k - s < r \leq 2^n - k - 3$, every positive integer is a sum of I values.

For $x = 0, 1, \dots, 3^n u - 2^n(uq + u - 1) - 1 = 2^n - u(2^n - r) - 1$,

$2^n(uq + u - 1) + x$ is a sum of $C = 2^n - u(2^n - r - k - 2) - 2$ values. The integers $\geq 3^nu$ and $\leq 2^n(uq + u) - 1 = 3^nu + u(2^n - r) - 1$ are sums of $D = u + u(2^n - r) - 1$ values. Then C and D are each $\leq 2^n - s$ if

$$(12) \quad \frac{s}{2^n - r - k - 2} \leq u \leq \frac{2^n - s}{2^n - r + 1}.$$

Increase r to $2^n - k - 3$ on the left and decrease r to $2^n - k - s + 1$ on the right to obtain the subinterval,

$$(13) \quad s \leq u \leq \frac{2^n - s}{k + s}.$$

Note that the greatest x is positive if $u \leq (2^n - 1)/(2^n - r)$; this follows from (12). There will exist an integer u satisfying (13) if the difference between the limits is ≥ 1 , and hence if

$$2^n > 2s + ks + s^2 + k.$$

Since $s < 3[(\frac{4}{3})^n]$, the above inequality is true if $2^n > 6(\frac{4}{3})^n + 3(\frac{4}{3})^nk + 9(\frac{16}{9})^n + k$. Multiply each side by $(\frac{9}{16})^n$ to get $(\frac{9}{8})^n > 6(\frac{3}{4})^n + 3(\frac{3}{4})^nk + 9 + k(\frac{9}{16})^n$. This holds for $n \geq 20$. Thus all integers in $(2^n(uq + u - 1), 2^n(uq + u - 1) + 2^n)$ are sums of $2^n - s$ values. It now follows from Lemma 10 that all integers $\geq 2^n(uq + u - 1)$ are sums of I values.

It will next be shown that the latter is true also of the integers in

$$(3^nw, 3^n(w + 1)) \text{ for } w = 1, 2, \dots, U - 1,$$

where U is the least integer satisfying (13) and hence is equal to s . Then it will have been proved that all integers $\geq 3^n$ are sums of I values. Since all $< 3^n$ are by Lemma 1 and the proof of Lemma 2 sums of I values, the proof of Lemma 14 will be complete.

The inequality

$$(14) \quad 2^n \geq wk + ws + s$$

will be assumed true for the present and later shown to be satisfied for $n \geq 35$. From (14) it follows that $2^n > w(k + s)$, whence

$$wr > w(2^n - k - s) > (w - 1)2^n, \quad w(2^n - r) < 2^n.$$

Consider now $J = (3^nw, 3^nw + 2^n)$ and the equations:

	Difference	Weight
$j\{-3^n + (q + 1)2^n\} = j(2^n - r)$	$2^n - r$	jk
		$(j = 0, 1, \dots, w - 1)$
$w\{-3^n + (q + 1)2^n\} = w(2^n - r)$	$2^n - w(2^n - r)$	wk .

To difference + weight $- 1$ add $w \cdot 3^n$ must be added to get decompositions. For $j \leq w - 1$ the largest sum is $F = 2^n - r + (w - 1)(k + 1)$. For

the final equation the sum is $G = 2^n - w(2^n - r) + w(k + 1) - 1$. Then $G \geq F$ if $(w + 1)r \geq 2^n w - k$. This follows from (14) and a hypothesis in Lemma 14. Hence every integer in the interval J is a sum of G values. Extend J to be $K = (3^n w, 3^n(w + 1))$ by adding $2^n x$, $1 \leq x \leq q$, to each integer in J . Then all in K are sums of $G + k + 1$ values. If

$$(15) \quad w(2^n - r - k - 1) \geq 1,$$

$G + k + 1 \leq I$. From a hypothesis in Lemma 14, $2^n - r - k - 1 \geq 2$, and (15) is satisfied.

It remains to show that (14) is true for $n \geq 35$. Now $w + 1 \leq U = s < 3(\frac{4}{3})^n$. Hence (14) will follow from $2^n > 3k(\frac{4}{3})^n + 9(\frac{16}{9})^n$. Multiply each side by $(\frac{9}{16})^n$. Then $(\frac{9}{8})^n > 3k(\frac{3}{4})^n + 9$. This holds for $n \geq 20$.

Lemmas 11 to 14 yield

THEOREM 3. *If $n \geq 35$ and $r \leq 2^n - k - 3$, every positive integer is a sum of $I = 2^n + k - 1$ values.*

Let $r \geq 2^n - k - 2$. If $r = 2^n - 1$, $3^{2^n} \equiv 1 \pmod{2^n}$. Since 3 belongs to the exponent 2^{n-2} modulo 2^n , $2n$ would be divisible by 2^{n-2} , and $n \leq 4$. If $r = 2^n - 3$, $3^{n-1} = (2^n(q + 1)/3) - 1$, whence $n \leq 5$. Since r is odd, it then follows that $r \leq 2^n - 5$. Thus

$$(16) \quad 5 \leq S \leq k + 2, \quad S = 2^n - r.$$

Ascent will be made from an interval containing 4^n . Let

$$(17) \quad 4^n = 3^n f + 2^n h + j, \quad 0 \leq 2^n h + j < 3^n, \quad 0 \leq j < 2^n.$$

Thus $f = [(\frac{4}{3})^n]$. To determine h and j in terms of q , r , and f , replace in (17) 3^n by $2^n q + r$. Then $j + fr \equiv 0$, $j - fS \equiv 0 \pmod{2^n}$. But $fS \leq f(k + 2) < fq < (\frac{4}{3} \cdot \frac{3}{2})^n$. Hence fS and j are numbers x for which $0 \leq x < 2^n$. Thus $|j - fS| < 2^n$ and $j = fS$. Then

$$(18) \quad 4^n = 2^n(fq + h + f), \quad h = 2^n - fq - f.$$

If $E_1 = \max(A = 2^n - f(q - S) - 1, 2^n - fS)$, every integer in the interval $(3^n f + 2^n h, 3^n f + 2^n(h + 1))$ is a sum of E_1 values as shown by

$$\begin{aligned} 3^n f + 2^n h + x & \quad (x = 0, 1, \dots, fS - 1), \\ 4^n + y & \quad (y = 0, 1, \dots, 2^n - fS - 1). \end{aligned}$$

By Lemma 9 and the above all integers in $(4^n, (n + 1)^n)$ are sums of $E_1 + k + 1 + [(\frac{4}{3})^n] + \dots + [((n + 1)/n)^n]$ values. Let $n \geq 35$ and t be the least integer such that $t > M^{-1}n(3684214 + 6 \log_{10} n)$.

Take $E_1 = 2^n - fS$. Then all integers $\geq 4^n$ are sums of

$$(19) \quad t + 2^n - fS + k + 1 + f + X$$

values,

$$X = [(\frac{5}{4})^n] + \dots + \left[\left(\frac{n+1}{n} \right)^n \right] < (n - 3)(\frac{5}{4})^n.$$

If $S - 1 \geq (t + 2)(\frac{3}{4})^n + (n - 3)(\frac{15}{16})^n$, (19) is $\leq I$. The right side decreases as n increases if $n > 15$. For $n \geq 35$, this inequality gives $S \geq 4.37704$. Since $S \geq 5$, the number of values is $\leq I$.

Next let $E_1 = A$. Then all integers $\geq 4^n$ are sums of $t + A + k + 1 + f + X$ values. This will be $\leq I$ if $f(q - S - 1) \geq t + X + 1$, and hence for $n \geq 35$ if $S \leq q - 6$. But $S \leq k + 2 < q - 6$. This proves

THEOREM 4. *If $n \geq 35$ and $r \geq 2^n - k - 2$, every integer $\geq 4^n$ is a sum of $I = 2^n + k - 1$ values.*

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THE WARING PROBLEM WITH SUMMANDS $1 + bx^n$.*

By MARY HABERZETLE.

A positive integer will be said to be represented by $[u, w]$ if it is a sum of u positive integers of the form $1 + bx^n$ or 0 and w positive integers of the form $a(1 + bx^n)$ or 0, where a and b are fixed positive integers. In this paper u and w will be determined such that for $19 \leq n \leq 400$, $1 \leq a \leq 4n$, $1 \leq b \leq 2n + 1$ every positive integer is represented by $[u, w]$. The problem is a modified Waring problem and was suggested by L. E. Dickson. The results obtained are stated in Theorem 3.

For brevity define $p_0 = 0$, $p_1 = 1$, $p_2 = 1 + b$, $p_3 = 1 + b \cdot 2^n, \dots, p_x = 1 + b(x - 1)^n$. For $v > 0$ denote by v' the number of terms, 0, 1, $1 + b$, required to represent v , and by v'' the number of terms, 0, 1, $1 + b$, required to represent each of $1, \dots, v$. For $v = 0$, define $v' = v'' = 0$. The notation $[x]$ is used to denote the integral part of x .

1. Lemmas for ascent. Lemmas 1, $\dots$, 4 have been proved by L. E. Dickson.¹

LEMMA 1. If $z \geq a$ and if all integers in the interval $(E, E + zp_k)$ are represented by $[A, B]$, then all in $(E, E + (z + av)p_k)$ are represented by $[A, B + v]$.

LEMMA 2. If $z \geq 1$ and all integers in $(E, E + zp_k)$ are represented by $[A, B]$, then all in $(E, E + (z + w)p_k)$ are represented by $[A + w, B]$.

LEMMA 3. Let R_j denote the least positive or zero residue modulo a of $G_j = [p_j/p_{j-1}]$, $j > 1$. Let $Q_j = (G_j - R_j)/a$. If all integers in $(E, E + ap_{m-1})$ are represented by $[R, S]$, then all in $(E, E + p_m)$ are represented by $[R, S + Q_m]$.

LEMMA 4. If $m \geq 4$ and if all integers in $(E, E + ap_3)$ are represented by $[A, B]$, then all in $(E, E + p_m)$ are represented by $[C, D]$, $C = A + (m - 4)(a - 1)$, $D = B + Q_4 + \dots + Q_m$.

LEMMA 5. Let C be a positive integer and define c to be the least positive residue of C modulo a . Each of the integers $0, 1, \dots, C - 1$ is represented by

* Received June 9, 1938.

¹ "Universal forms $\Sigma a_i x_i^n$ and Waring's problem," *Acta Arithmetica*, vol. 2 (1937), pp. 178-179.

$$[(c-1)'', k'] \quad \text{or} \quad [(a-1)'', (k-1)''],$$

where $k = (C-c)/a$. The second form is to be suppressed if $k = 0$.

Those integers which are $\geq C-c = ka$ are sums of ka and $0, 1, \dots, c-1$. The latter are sums of $(c-1)''$ terms $0, 1, 1+b$, and ka is a sum of k' terms $0, a, a(1+b)$.

Only when $k \geq 1$ are there further integers, $ka - ma + j$ ($j = 0, 1, \dots, a-1$; $k \geq m \geq 1$), and these are represented by $[(a-1)'', (k-1)'']$.

LEMMA 6. Let $p_4 = qp_3 + r$, $0 \leq r < p_3$. Denote by f , g , and d the least positive residues modulo a of $p_3 - r$, r , and q , respectively. Then every integer in the interval $(qp_3, (q+1)p_3)$ is represented by one of

$$(1) \quad \left[d + 1 + (g-1)'', \left(\frac{r-g}{a} \right)' + \left(\frac{q-d}{a} \right) \right],$$

$$(2) \quad \left[d + (a-1)'', \left(\frac{r-g}{a} - 1 \right)'' + \left(\frac{q-d}{a} \right) \right],$$

$$(3) \quad \left[(f-1)'' + 1, \left(\frac{p_3 - r - f}{a} \right)' \right],$$

$$(4) \quad \left[(a-1)'' + 1, \left(\frac{p_3 - r - f}{a} - 1 \right)'' \right].$$

The second is to be suppressed if $r \leq a$ and the fourth if $p_3 - r \leq a$.

By Lemma 5, $0, 1, \dots, r-1$ are represented by $\left[(g-1)'', \left(\frac{r-g}{a} \right)' \right]$ if $r \leq a$, but by it or $\left[(a-1)'', \left(\frac{r-g}{a} - 1 \right)'' \right]$ if $r > a$. To these add $\left[d, \frac{q-d}{a} \right]$ which represents $qp_3 = dp_3 + \frac{q-d}{a} \cdot ap_3$. The weight of (1) is increased by one for reasons which will appear shortly.

The remaining integers in the interval (except the final one) are sums of $qp_3 + r = p_4$ and $0, 1, \dots, p_3 - r - 1$. These are represented by $\left[(f-1)'' + 1, \left(\frac{p_3 - r - f}{a} \right)' \right]$ or $\left[(a-1)'' + 1, \left(\frac{p_3 - r - f}{a} - 1 \right)'' \right]$.

Finally, $(q+1)p_3$ is represented by $\left[d + 1, \frac{q-d}{a} \right]$ and hence by (1).

LEMMA 7. Every integer in the interval $(qp_3, (q+a)p_3)$ is represented by one of the forms obtained by adding $a-1$ to the first entries of the forms (1), $\dots$, (4).

The lemma follows from Lemmas 2 and 6.

2. Representation of integers $\leq qp_3 - 1$.

LEMMA 8. *Let $a < p_3$. Let c and d be the least positive residues modulo a of p_3 and q , respectively. Then every positive integer $\leq qp_3 - 1$ is represented by one of the two forms,*

$$\begin{aligned}[U_0, V_0] &= \left[a - 1 + (c - 1)'', \left(\frac{p_3 - c}{a} \right)' + \frac{q - d}{a} \right], \\ [U_1, V_1] &= \left[a - 1 + (a - 1)'', \left(\frac{p_3 - c}{a} - 1 \right)'' + \frac{q - d}{a} \right].\end{aligned}$$

The integers in question are $xp_3 + y$, $0 \leq x \leq q - 1$, $0 \leq y \leq p_3 - 1$. By Lemma 5 the y 's are represented by

$$\left[(c - 1)'', \left(\frac{p_3 - c}{a} \right)' \right] \quad \text{or} \quad \left[(a - 1)'', \left(\frac{p_3 - c}{a} - 1 \right)'' \right].$$

Since $q - 1 \equiv d - 1 \pmod{a}$ and $d \leq a$, xp_3 is represented by $[a - 1, (q - d)/a]$. Hence all are represented by $[U_0, V_0]$, $[U_1, V_1]$.

THEOREM 1. *Let $1 \leq a \leq 4n$, $1 \leq b \leq 2n + 1$. Let c and d be the least positive residues modulo a of p_3 and q , respectively. Then every positive integer $\leq qp_3 - 1$ is represented by*

$$(5) \quad H = \left[4n + a(2n + 1) + (a - 1)'' - 1, \left(\frac{p_3 - c}{a} \right)'' + \frac{q - d}{a} - [4n/a] - 2n \right].$$

In Lemma 8 increase U_0 to U_1 . Both $\left(\frac{p_3 - c}{a} \right)'$ and $\left(\frac{p_3 - c}{a} - 1 \right)''$ are less than or equal to $\left(\frac{p_3 - c}{a} \right)''$. It is a consequence then of Lemma 8 that every positive integer $\leq qp_3 - 1$ is represented by

$$F = \left[a - 1 + (a - 1)'', \left(\frac{p_3 - c}{a} \right)'' + \frac{q - d}{a} \right].$$

In F replace each of $[4n/a] + 2n$ terms ap_x by $p_x + \dots + p_x$ (with a summands) to obtain H . It will be shown later that

$$(6) \quad \left(\frac{p_3 - c}{a} \right)'' + \frac{q - d}{a} - [4n/a] - 2n \geq 0.$$

3. Formulas for t ascents at one step. No proofs will be given for the lemmas in this section. References are made to similar lemmas and their proofs.

LEMMA² 9. If $g \geq 0$ and $s \geq g + a$, there exists a positive integer i such that

$$g \leq s - a(1 + bi^n) < g + n(ab s^{n-1})^{1/n}.$$

LEMMA³ 10. Let g be an integer ≥ 0 , L an integer $\geq a$,

$$G = \{(ab)^{-1}(L/n)^n\}^{1/(n-1)}.$$

If $G \geq g + L$ and if all integers between g and $g + L$ inclusive are represented by $F = [R, S]$, then every integer s between g and G inclusive is represented by $F_1 = [R, S + 1]$.

Write $L_0 = g + L$, $v = (1 - g/L_0)/n$, $L_1 = G$. Lemma 10 becomes

LEMMA 11. Let g be an integer ≥ 0 , $L_0 \geq g + a$,

$$L_1 = \{(ab)^{-1}(L_0 v)^n\}^{1/(n-1)}.$$

If $L_1 \geq L_0$ and if all integers between g and L_0 inclusive are represented by $F = [R, S]$, then all between g and L_1 inclusive are represented by $F_1 = [R, S + 1]$.

LEMMA⁴ 12. Let g be an integer ≥ 0 , $v = (1 - g/L_0)/n$, $L_0 \geq g + a$, $L_0 v^n \geq ab$. Compute L_t from

$$\log_{10} L_t = \left(\frac{n}{n-1} \right)^t (\log_{10} L_0 + w) - w,$$

$w = n \log_{10} v - \log_{10} ab$. If all integers between g and L_0 inclusive are represented by $F = [R, S]$ then all between g and L_t inclusive are represented by $F_t = [R, S + t]$.

Apply the above lemma with $a = 1$ and L_t in place of L_0 to obtain

LEMMA 13. Let g be an integer ≥ 0 , $V = (1 - g/L_t)/n$, $L_t \geq g + 1$, $L_t V^n \geq b$. If all integers between g and L_t inclusive are represented by $F_t = [R, S + t]$ then all between g and L_{t+T} are represented by $F_{t+T} = [R + T, S + t]$ and

$$(\gamma) \quad \log_{10} L_{t+T} = \left(\frac{n}{n-1} \right)^{t+T} (\log_{10} L_0 + w) + \left(\frac{n}{n-1} \right)^T (W - w) - W,$$

where $W = n \log_{10} V - \log_{10} b$.

² L. E. Dickson, "Generalizations of Waring's theorem on fourth, sixth, and eighth powers," *American Journal of Mathematics*, vol. 49 (1927), p. 242.

³ Dickson, "Universal forms $\Sigma a_i x_i^n$ and Waring's problem," *op. cit.*, pp. 184-185.

⁴ *Ibid.*, p. 185.

Lemma 13 will be used for L_t so large that $1/n$ is a sufficiently close approximation to V , whence $W = -n \log_{10} n - \log_{10} b$. But $v < 1/n$, whence $W - w > \log_{10} a$. Since the last two terms of (7) are positive

$$(8) \quad \log_{10} L_{t+T} > \left(\frac{n}{n-1} \right)^{t+T} (\log_{10} L_0 + n \log_{10} v - \log_{10} ab).$$

4. Results from the asymptotic theory and prime number theory. The following theorem is a consequence of a lemma⁵ proved by L. E. Dickson.

THEOREM 2. *Let k_0 be the least integer exceeding $(\log r_1)/(-\log(1-v))$, where $r_1 = n^6(6n-1)/(n-d)$, $d = 1 + 2n^2z$, $z = v^3/12$ and $v = 1/n$. Take $k_1 = 2k_0$. Then if $1 \leq a \leq 4n$, $n \geq 9$, every integer $\geq N_1$ is represented by $[4n + b - 1, 3k_1 - 2]$ where $N_1 = bN + 4n + a(3k_1 - 2)$ and $\log_{10} N = .8n^6$.*

LEMMA⁶ 14. *If i is a positive integer and $K_i = [i^n/(i-1)^n]$, $m \geq 6$, $n \geq 9$, then $\sum_{i=4}^m K_i < X$, $X = K_4 + 5K_5 + m - n - 9 + n \log(m-1)/4 - \left(\frac{n}{2}\right)/(m-1)$.*

5. Solution for $19 \leq n \leq 400$.

THEOREM 3. *Let $1 \leq a \leq 4n$ and $1 \leq b \leq 2n + 1$. Let $19 \leq n \leq 400$ and denote by c and d the least positive residues modulo a of p_3 and q , respectively. Then every positive integer is represented by*

$$H = \left[4n + a(2n + 1) + (a-1)'' - 1, \right. \\ \left. \left(\frac{p_3 - c}{a} \right)'' + \frac{q - d}{a} - [4n/a] - 2n \right].$$

For integers $\leq qp_3 - 1$ the theorem is true by Theorem 1. For integers $\geq N_1$ it is true by Theorem 2 since

$$b \leq 2n + 1 \leq a(2n + 1) + (a-1)'',$$

and

$$(9) \quad \left(\frac{p_3 - c}{a} \right)'' + \frac{q - d}{a} - [4n/a] - 2n \geq 3k_1 - 2, \quad n \geq 19.$$

The latter inequality is obtained by noting that

$$\left(\frac{p_3 - c}{a} \right)'' \geq (2^n + 1)/2a - 1/2 \quad \text{and} \quad k_1 \leq 4n \log n + 2n \log 6.$$

⁵ *Ibid.*, p. 190.

⁶ *Ibid.*, p. 188.

As a consequence of these inequalities (9) is implied by

$$(10) \quad 1 \geq (96n^2 \log n + 48n^2 \log 6 + 8n + 16n^2)/2^n.$$

For $n = 17$, (10) is true, and since the right-hand side decreases as n increases it holds for $n \geq 17$.

Since $3k_1 - 2 > 0$, it is now seen that (6) is true.

It remains then to prove Theorem 3 for integers $\geq qp_3$ and $< N_1$. This is done by ascent. From Lemma 4 and the first part of Lemma 7 all integers in the interval $J = (qp_3, qp_3 + q_{m+1})$ are represented by $[K, D]$, where $K = A + (m-2)(a-1)$ and $D = B + \sum_{i=4}^{m+1} Q_i$, while $[A, B]$ was initially one of the forms (1), $\dots$, (4) but may be replaced by one of

$$(11) \quad \left[d + 1 + (a-1)'', \left(\frac{r-g}{a} \right)'' + \frac{q-d}{a} \right], \\ \left[(a-1)'' + 1, \left(\frac{p_3 - r - f}{a} \right)'' \right].$$

Apply Lemmas 12, 13 with $g = p_4$, $L_0 = p_{m+1}$, both in J . The conditions, $L_0 v^n \geq ab$, $L_t V^n \geq b$, will later be seen to hold after m is restricted. Let Z be the least integral value of $t + T$ for which the second member of (8) exceeds $\log_{10} N_1$, where N_1 is the constant in Theorem 2. It will be shown that $[T + K, t + D] \leq H$.

For the first case from (11), let

$$(12) \quad T = 4n + a(2n + 1) - d - 2 - (a-1)(m-2) - 2a.$$

Then

$$t + \left(\frac{r-g}{a} \right)'' + \sum_{i=4}^{m+1} Q_i \leq \left(\frac{p_3 - c}{a} \right)'' - [4n/a] - 2n.$$

Multiply both sides of the inequality by a , substitute $aZ - aT$ for at , and then replace T by (12). This gives

$$aZ - 4an - a^2(2n + 1) + ad + 2a + a(a-1)(m-2) \\ + 2a^2 + a \left(\frac{r-g}{a} \right)'' + a \sum_{i=4}^{m+1} Q_i \leq a \left(\frac{p_3 - c}{a} \right)'' - a[4n/a] - 2an.$$

Increase $\left(\frac{r-g}{a} \right)''$ to $(r-g)/a(1+b) + b$ and $[4n/a]$ to $4n/a$. Then

$$\frac{r-g}{1+b} \leq -aZ + 4an + a^2(2n + 1) - ad - 2a - 2a^2 \\ (13) \quad -a(a-1)(m-2) - ab - a \sum_{i=4}^{m+1} Q_i + (1+b \cdot 2^n)(1+b) \\ - 4n - 2an - c/(1+b).$$

Write $E = aZ - 4an - a^2(2n + 1) + 2a + 2a^2 + a(a - 1)(m - 2) + ab + a \sum_{i=4}^{m+1} Q_i + c/(1 + b) + 4n + 2an$. In (13) decrease g to 1. Then

$$(14) \quad (r - 1)/(1 + b) \leq -E - ad + (1 + b \cdot 2^n)/(1 + b).$$

Similarly, for the second case from (11) let

$$(15) \quad T = 4n + a(2n + 1) - 2 - (a - 1)(m - 2) - 2a.$$

Proceeding as above, one obtains

$$(16) \quad (r + 1)/(1 + b) \geq E - q + d.$$

Let $m = 2n$. Then for $L_0 = p_{m+1} = 1 + b(2n)^n$, $L_0 v^n > ab$. For, when $n \geq 19$, $(1 + b(2n)^n)v^n$ is approximately equal to $(1 + b(2n)^n)/n^n > 2^n b$. Now $2^n b > 4nb \geq ab$ when $n \geq 5$. It follows from this that $L_t V^n > b$. The conditions in Lemmas 12 and 13 are therefore satisfied.

For T in (12) and (15) it will now be shown that $t = Z - T \geq 0$. For this purpose Z will be determined such that $\log_{10} L_{t+T} > \log_{10} N_1$. It suffices to take $\log_{10} N_1 = n^6$. In (8) take $v = 1/n$, decrease $L_0 = 1 + b(2n)^n$ to $b(2n)^n$ and increase a to $4n$, thereby increasing $t + T$. Then

$$\left(\frac{n}{n-1}\right)^{t+T} (n \log_{10} 2 - \log_{10} 4n) > n^6,$$

or

$$(17) \quad Z = t + T > \frac{6 \log_{10} n - \log_{10} (n \log_{10} 2 - \log_{10} 4n)}{\log_{10} n - \log_{10} (n - 1)}.$$

From (12) and (15) it is seen that $T > 0$ and $< 10n$. The numerator in (17) is $> 5 \log_{10} n$. It is therefore sufficient to show that

$$t + T > \frac{5 \log_{10} n}{\log_{10} n - \log_{10} (n - 1)} > 10n.$$

By computation this inequality may be shown to be satisfied for $19 \leq n \leq 400$.

When $19 \leq n \leq 400$ the two conditions (14) and (16) are satisfied. Increase E to

$$(18) \quad E_1 = 4nZ + 22n - 1 + a \sum_{i=4}^{2n+1} Q_i + 8n^2.$$

In (14) increase $r - 1$ to r and a to $4n$, multiply both sides by $1 + b$ and divide by $1 + 2^n b$. Then (14) holds if

$$r/(1 + 2^n b) \leq 1 - (E_1 + 16n^2)(1 + b)/(1 + 2^n b).$$

Similarly, (16) holds if

$$r/(1+2^nb) \geq (E_1 + 4n)(1+b)/(1+2^nb).$$

Now

$$(1+b)/(1+2^nb) \leq 2/(1+2^n).$$

The two inequalities above will therefore hold if

$$(19) \quad \frac{2E_1 + 8n}{1 + 2^n} \leq \frac{r}{1 + 2^nb} \leq 1 - \frac{2(E_1 + 16n^2)}{1 + 2^n}.$$

The decimal part of $(1+3^nb)/(1+2^nb)$ is $r/(1+2^nb)$;

$$(1+3^nb)/(1+2^nb) = (3/2)^n - ((3/2)^n - 1)/(1+2^nb).$$

The latter fraction decreases as n and b increase. When $n = 19$ it is at most .004⁺, $n = 20$ at most .003⁺. Hence the decimal part of $(1+3^nb)/(1+2^nb)$ agrees with that of $(3/2)^n$ to an increasing number of decimal places.

Let $n = 19$. From (17), Z may be taken to be 302. From Lemmas 3 and 14,

$$\begin{aligned} a \sum_{i=4}^{2n+1} Q_i &\leq \sum_{i=4}^{2n+1} G_i = [p_4/p_3] + \cdots + [p_{2n+1}/p_{2n}] \leq [(3/2)^n] + [(4/3)^n] \\ &+ \cdots + [(2n/2n-1)^n] = [(3/2)^n] + \sum_{i=4}^{2n} K_i < [(3/2)^n] + X; \end{aligned}$$

$X < 634.645589$. Also, $[(3/2)^{19}] \leq 2217$. Hence, $E_1 \leq 28708.645589$. From (19), $.111330 \leq r/(1+2^nb) \leq .866925$. Since $1 - 2(E_1 + 16n^2)/(1+2^n)$ increases with n and $(2E_1 + 8n)/(1+2^n)$ decreases when n increases, Theorem 3 will be true for any $n \geq 19$ for which $r/(1+2^nb)$ lies between the above limits. The condition is satisfied for $n = 19, 20$.

Let $n = 20$. Then $Z = 323$, $X < 802.08$, $[(3/2)^{20}] \leq 3326$, and $E_1 \leq 33607.08$. Hence, $.064253 \leq r/(1+2^nb) \leq .923692$. This holds for $n = 20, 21, \dots, 28$.

Let $n = 29$. Now $Z = 522$, $X < 7520.92208$, $[(3/2)^{29}] \leq 127840$, and $E_1 \leq 203277.92208$. Hence $.000758 \leq r/(1+2^nb) \leq .9991925$. This is true for $29 \leq n \leq 400$.

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